$$T_4 = \int_0^1 \sin\psi \sin\zeta \sinh\xi dx$$

$$\psi = \mu\sqrt{1 - x^2}$$

$$\psi_1 = \mu x$$

$$\zeta = \nu x$$

$$\zeta_1 = \nu\sqrt{1 - x^2}$$

$$\xi = \rho x$$

$$\xi_1 = \rho\sqrt{1 - x^2}$$

$$\mu = k_1 r_0$$

$$\nu = k_2 r_0 \sin\theta$$

$$\rho = \alpha r_0 \cos\theta.$$

ACKNOWLEDGMENT

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An Equivalent Circuit for the "Centipede" Waveguide

T. M. REEDER

Abstract—An equivalent circuit is presented for the "centipede" coupled-cavity waveguide. The "centipede" waveguide, which is typical of the class of slow-wave structures suitable for use in a wide-band high-power traveling-wave tube, has two pass bands which may interact strongly with a small-diameter electron beam. The equivalent circuit is able to represent both of these pass bands. A detailed comparison with an S-band "centipede" waveguide shows that the equivalent circuit can represent the dispersion, interaction, and loss characteristics of the waveguide within a few percent.

I. INTRODUCTION

THE COUPLED-CAVITY waveguide has found extensive use in recent years as a slow-wave circuit in high-power, broadband traveling-wave tubes. Coupled-cavity structures that use resonant coupling elements are a logical choice for high-power tube use because of their rugged construction, relatively high interaction impedance, and large cold bandwidth, often greater than 30 percent [1]. However, analytical studies of tubes using these waveguides are usually tedious because exact electromagnetic field solutions are difficult, even impossible, to obtain. The analysis may be made much simpler if an equivalent circuit can be found to represent the waveguide, but it is difficult to find a simple circuit which will accurately represent the propagation characteristics of broadband waveguides like the "cloverleaf" [1], the "long-slot" [2], and the "centipede" [3] over all operating frequencies.

Equivalent circuits given first by Pierce [4] and later used by Gould [5] and Collier et al. [6] provide an adequate representation of relatively narrow-band waveguides like the "space-harmonic" structure of Chodorow and Nalos [7]. However, the fact that these circuits do not include the effect of intercavity coupling resonance leads to considerable error when they are applied to broadband coupled-cavity waveguides where the cavity and coupling element resonant frequencies may be close together. Curnow [8], [9] has recently shown that by including the coupling element resonance, the dispersion and impedance properties of the "long-slot" waveguide can be accurately represented. Unfortunately, the equivalent circuit must correspond closely to the geometrical configuration of the waveguide as both Curnow and Gittens [10] have demonstrated. The resultant circuit may be quite complicated and difficult to analyze.

The purpose of this paper is to show that one coupled-cavity waveguide, the "centipede," can be accurately represented by a relatively simple equivalent circuit. Waveguides in the "centipede" class have two pass

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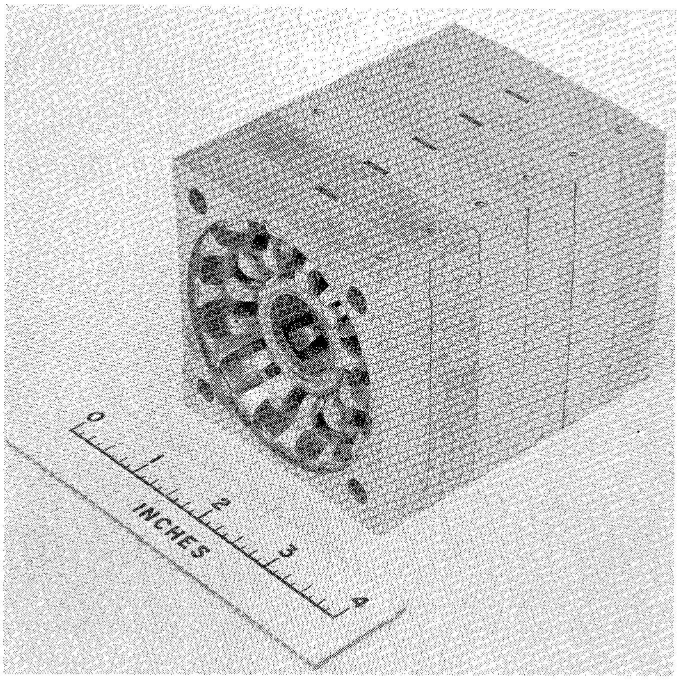

Fig. 1. A view of the S-band "centipede" waveguide. Distributed loss was added after this picture was taken.

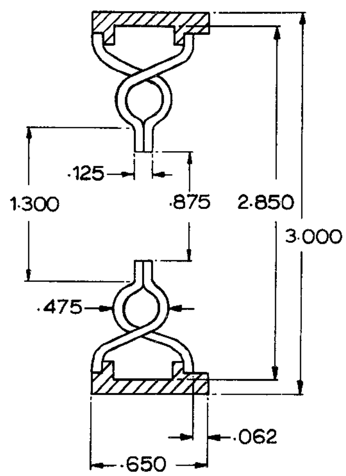

Fig. 2. Cross-sectional drawing of a periodic section of the S-band "centipede" waveguide.

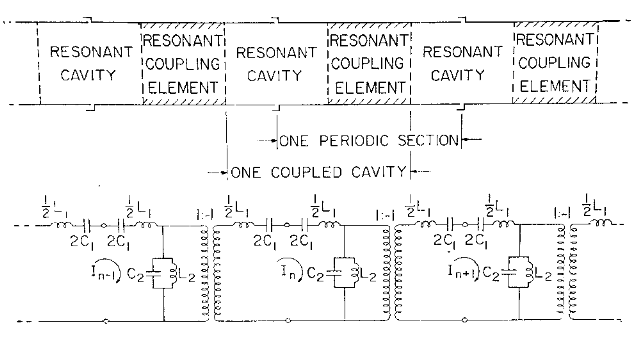

Fig. 3. The equivalent circuit.

bands with field distribution similar to the TM_{01} circular waveguide mode. Both of these pass bands are important for traveling-wave tube operation, since the TM_{01} field configuration interacts strongly with a small-diameter electron beam. The equivalent circuit discussed here is shown to be an accurate model of the propagation characteristics for both of these pass bands. The dispersion, interaction, and loss characteristics of the circuit are derived and compared with experimental data for an S-band "centipede" waveguide [11].

II. The Equivalent Circuit

The "centipede" waveguide may be described as a chain of pillbox resonators in which the fields of one cavity are heavily coupled to those of the next by a resonant loop diaphragm. These loops, which are shown in Figs. 1 and 2, provide negative mutual coupling in that if two adjacent cavities have the same orientation of transverse magnetic field, the fields will tend to cancel each other over the volume of the loops.

Let us consider the partial schematic diagram shown in Fig. 3. One might expect that an array of resonant cavities coupled together by resonant loops could be represented by an iterative equivalent circuit containing two resonant circuits, one representing the cavity resonance and the other modeling the coupling-loop resonance. In Fig. 3 the series resonant circuit L_1-C_1 represents the cavity resonance, and C_1 is the effective capacitance that a single cavity would present to an electron beam shot through the axis of the cavity chain. The resonance of the coupling loops is modeled by the parallel circuit L_2-C_2, and the negative mutual effect of the loops is provided by the polarity inverting transformer. The effect of waveguide attenuation may be included by adding loss to the two resonant circuits, as is discussed in Section V. Since the equivalent circuit is to represent waveguide modes with a TM_{01} field distribution, the loop current at each circuit cavity I_n is, by analogy, proportional to the current flowing in the wall of the corresponding waveguide cavity.

III. Circuit Dispersion

A dispersion equation may be derived for the equivalent circuit by studying the possible modes that may propagate over an infinite set of cavities. We shall assume that the propagation of modes may be represented by

$$I_{n+1} = e^{-\Gamma} I_n \tag{1}$$

where Γ is a complex propagation constant given by

$$\Gamma = \xi + j\theta. \tag{2}$$

By writing down the mesh equations for a single coupled cavity, one finds that the circuit dispersion is given by

$$-\cosh \Gamma = 1 + \frac{Z_1}{2Z_2} \tag{3}$$

where Z_1 and Z_2 are, respectively, the series impedance of L_1-C_1 and the shunt impedance of L_2-C_2.

We note that for frequencies in a pass band $\xi=0$ if the circuit is lossless, and if we expand Z_1 and Z_2, the dispersion equation becomes

$$f^4 - f^2[f_a^2 + f_b^2 + 2kf_a^2(1 + \cos\theta)] + f_a^2 f_b^2 = 0 \quad (4)$$

where it has been convenient to define

$$f_a = \frac{1}{2\pi\sqrt{L_1C_1}} = \text{cavity resonant frequency} \quad (5)$$

$$f_b = \frac{1}{2\pi\sqrt{L_2C_2}} = \text{coupling loop resonant frequency} \quad (6)$$

$$k = C_1/C_2 = \text{coupling parameter.} \quad (7)$$

From (4), it is clear that the circuit has four cutoff frequencies which may define two pass bands. One may easily show that the two pass bands will look like those shown in the Brillouin diagram, Fig. 4. The fact that the lower frequency pass band is forward wave fundamental and the higher frequency pass band is backward wave fundamental is expected from the analysis of Chodorow and Craig [1] and is substantiated by the experimental results of Gittens et al. [12] and Pearce [3] for negative mutual coupled waveguides.

It will be instructive to consider the relations between the four circuit cutoff frequencies defined in Fig. 4 and the circuit constants f_a, f_b, and k. The following equations are deduced by solving (4) at $\theta=0$, π:

$$f_1^2 + f_4^2 = f_a^2(1+4k) + f_b^2 \quad (8)$$

$$f_1^2 \cdot f_4^2 = f_a^2 \cdot f_b^2 \quad (9)$$

$$f_2^2 + f_3^2 = f_a^2 + f_b^2 \quad (10)$$

$$f_2^2 \cdot f_3^2 = f_a^2 \cdot f_b^2. \quad (11)$$

Equations (8) to (11) are sufficient to determine f_a, f_b, and k from the four circuit cutoff frequencies. However, we note from (9) and (11) that

$$f_1 \cdot f_4 = f_2 \cdot f_3. \quad (12)$$

Thus, only three of the four cutoff frequencies may be selected arbitrarily, the fourth being determined by (12).

If the cavity and coupling-loop resonant frequencies are not equal, one can see from (10) and (11) that either

$$f_a = f_2 \qquad f_b = f_3 \quad (13)$$

or

$$f_b = f_2 \qquad f_a = f_3. \quad (14)$$

We conclude that only one of the two circuit resonances may occur in each pass band and that one of these resonances occurs at the upper frequency cutoff of the lower pass band while the other is seen at the lower cutoff of the higher frequency pass band. Since only one of the resonances occurs in each pass band, the pass bands are frequently referred to as the "cavity" pass band and the "coupling-loop" pass band.

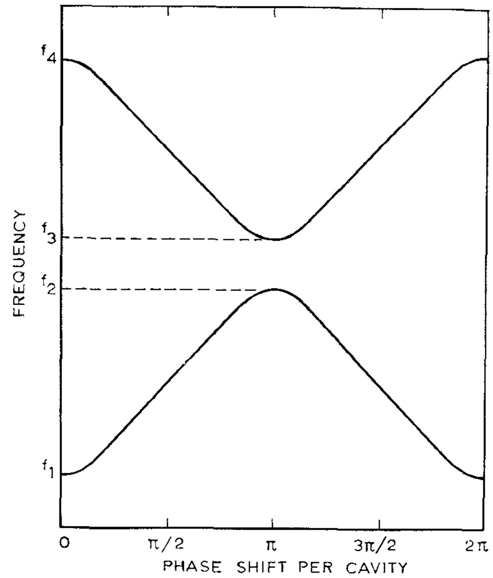

Fig. 4. A typical Brillouin diagram for the equivalent circuit.

Suppose we turn our attention to the constant k. From (8) and (10) we find

$$k = \frac{(f_4^2 - f_3^2) - (f_2^2 - f_1^2)}{4f_a^2} \quad (15)$$

which shows that k is related to the bandwidths of the two pass bands. Although it is not obvious from (15), one can show that k increases as the pass band bandwidths become greater.

We are now prepared to compare the dispersion of the equivalent circuit with the measured dispersion for the "centipede" waveguide shown in Fig. 1. The waveguide dispersion data, given in Fig. 5, is seen to have the same form as the equivalent circuit Brillouin diagram, Fig. 4. The waveguide was designed so that the coupling-loop resonant frequency was higher than the cavity resonance [11], and the lower pass band, which is the operating pass band for traveling-wave tube use, is the "cavity" pass band. Therefore, the circuit constants f_a and f_b are related to the cutoff frequencies f_2 and f_3 by (13). Being most concerned that the circuit provides an accurate model of the operating pass band, we shall select f_1 and f_2 to be equal to the operating pass band cutoff frequencies of the waveguide. The frequencies f_3 and f_4 are then selected so that (12) is satisfied, and the difference between the dispersion for the circuit and waveguide is small over the "loop" pass band. Having chosen the cutoff frequencies for the equivalent circuit, the constants f_a, f_b, and k can be computed from (13) and (15), and the circuit dispersion can be calculated from (4). The above method was used to compute the equivalent circuit dispersion shown in Fig. 5. The

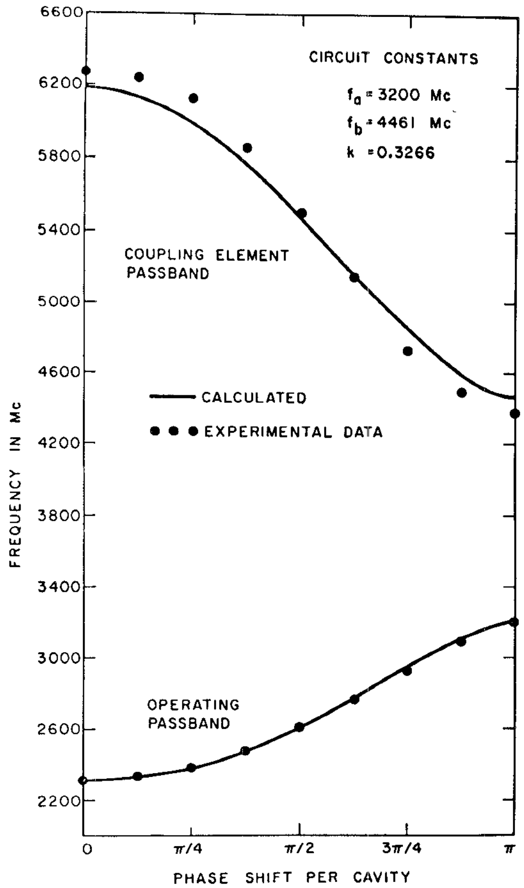

Fig. 5. A comparison of the Brillouin diagram calculated for the equivalent circuit with the experimentally measured data for the "centipede" waveguide.

computed dispersion agrees with the experimental data to within one-half percent over the operating pass band and to within two percent over the coupling-loop pass band.

IV. The Interaction Ratio

The ratio $|E_z|^2/W_l$ has been shown to be a useful quantity for comparing the relative worth of slow-waveguides for traveling-wave tube use [1]. Here E_z is the axial electric field of mode traversing the guide and W_l is the stored energy per unit length. For coupled-cavity waveguides it is convenient to define an analogous quantity $|V|^2/W$ where V is the voltage across one cavity and W is the energy stored per cavity. Hereafter, the quantity $|V|^2/W$ will be called the interaction ratio.

In Section III, only three constraints (f_a, f_b, and k) were placed upon the four energy storage elements of the equivalent circuit in determining the circuit dispersion. A fourth constraint will be provided here by forcing the value of the interaction ratio calculated for the equivalent circuit to match the value obtained experimentally for the "centipede" waveguide. The four circuit elements will then be uniquely defined.

The interaction ratio may be found experimentally by well-known perturbation methods [3], [13] in which a number of coupled-cavities are made resonant by the addition of shorting plates located at planes of symmetry. If a thin dielectric rod is introduced along the axis of the waveguide, the resonant frequency is shifted by an amount Δf, and the quantity $|E_z|^2/W$ is proportional to $\Delta f/f$. We must then define a suitable relation between V and E_z, over the cavity volume occupied by the electron beam. The author has concluded [11] that a suitable method is to define the cavity voltage as the product of the root mean average of E_z over the beam volume times the cavity gap length. The experimental interaction ratio then has the form

$$\frac{|V|^2}{W} \simeq \left|\frac{\Delta f}{f}\right| \frac{4L}{(\epsilon - \epsilon_0)\pi d^2} \tag{16}$$

where L is the periodic length, d is the radius of the perturbing rod, and ϵ and ϵ_0 are the permittivity of the rod and vacuo, respectively.

The equivalent circuit interaction ratio may be computed as follows. With reference to Fig. 3, the cavity voltage at the nth coupled-cavity is

$$V = I_n/j\omega C_1 \tag{17}$$

and the average stored energy in the nth cavity is given by

$$W = \frac{1}{4}\frac{dX_1}{d\omega}|I_n|^2 + \frac{1}{4}\frac{dX_2}{d\omega}|I_n + I_{n+1}|^2 \tag{18}$$

where X_1 and X_2 are the reactances of the cavity and coupling-loop resonant circuits. By combining (17) and (18) with the mode assumption (1) and the dispersion equation (3), the equivalent circuit interaction ratio is shown to be

$$\frac{|V|^2}{W} = \frac{2}{C_1}\frac{1 - f^2/f_b^2}{1 - f^4/f_a^2 f_b^2}. \tag{19}$$

The constants f_a and f_b were fixed when the circuit dispersion was specified, but C_1 is still arbitrary.

Equations (16) and (19) provide the means to match the interaction ratio of the equivalent circuit and experimental waveguide. We can select C_1 so that the right-hand sides of (16) and (19) are equal at all frequencies of interest. As a basis for comparison, the experimentally measured $|V|^2/W$ data for the operating pass band of the "centipede" waveguide is shown in Fig. 6 along with the computed curve for the equivalent circuit with C_1 arbitrarily chosen to be 3.8 pF. At first glance it would appear that the circuit does not accurately represent the waveguide interaction ratio over the entire pass band if C_1 is a constant. There are, however, two points to consider. The small signal gain of a traveling-wave tube is approximately equal to the cube root of the interaction ratio. The error introduced into a gain calculation is therefore about one third of the percentage by which the circuit fails to predict the waveguide $|V|^2/W$ data. We should also consider that, although one would like to make the circuit elements

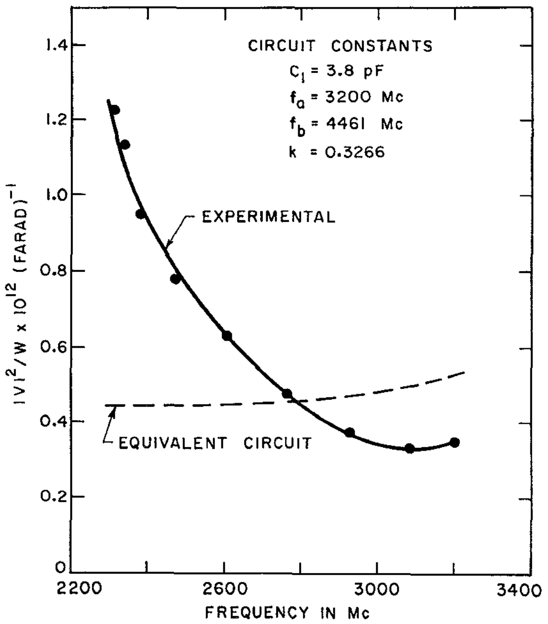

Fig. 6. A comparison of the interaction ratio computed for the equivalent circuit with experimentally measured data for the "centipede" waveguide.

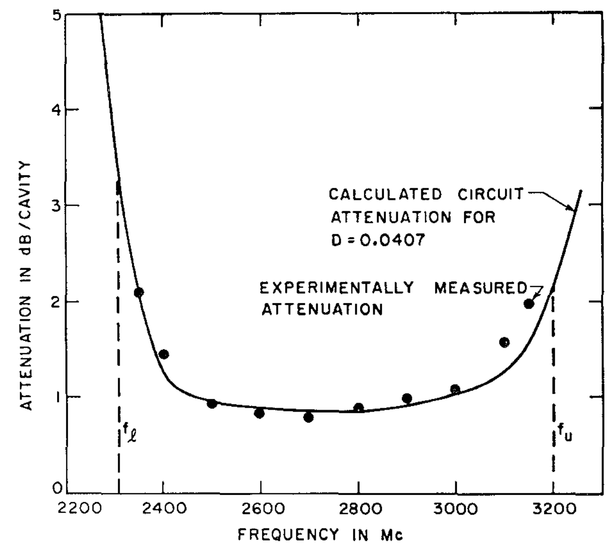

Fig. 7. A comparison of the circuit attenuation calculated for the equivalent circuit with the experimentally measured data for the "centipede" waveguide.

constant, the physical analogy given in Section II is not invalidated by allowing C_1 to be a function of frequency. By selecting C_1 at each frequency of interest, the equivalent circuit could predict the interaction ratio of the "centipede" waveguide over the entire operating pass band.

V. The Addition of Loss

The effect of distributed waveguide loss may be included in the equivalent circuit by adding loss elements to the resonant circuits $L_1 - C_1$ and $L_2 - C_2$. When loss is present, the resonant circuit impedances become

$$Z_1 = j\omega L_1(1 - jD_1) + \frac{1}{j\omega C_1} \tag{20}$$

$$Z_2 = \frac{1}{j\omega C_2(1 - jD_2) + \dfrac{1}{j\omega L_2}} \tag{21}$$

where D_1 and D_2 are the reciprocals of the resonant circuit Q factors. We shall assume that the loss is evenly divided between the two resonant circuits, that is,

$$D_1 = D_2 = D. \tag{22}$$

If the above expressions are substituted into (3), the circuit dispersion equation, we find

$$\cosh \xi \cos \theta = \frac{f^2(1 - D^2) + f_a^2 f_b^2/f^2 - f_a^2(1 + 2k) - f_b^2}{2kf_a^2} \tag{23}$$

$$\sinh \xi \sin \theta = \left(\frac{D}{2k}\right)\frac{f_a^2 + f_b^2 - 2f^2}{f_a^2}. \tag{24}$$

Equations (23) and (24) may be solved for ξ and θ once D and f are known.

If the circuit is not excessively lossy, approximate solutions of (23) and (24) may be found. Suppose that D is 0.05 or less (circuit Q's of 20 or greater). Then ξ will also be small so that the small argument approximations of sinh ξ and cosh ξ may be used. With small error in the middle of a pass band, (23) may be replaced by (4), the lossless dispersion equation, and (24) becomes

$$\xi \cong \left(\frac{D}{2k}\right)\frac{f_a^2 + f_b^2 - 2f^2}{f_a^2 \sin \theta}. \tag{25}$$

Similarly, one can verify that at a cutoff frequency

$$\xi \cong \sqrt{\left(\frac{D}{2k}\right)\frac{|f_a^2 + f_b^2 - 2f^2|}{f_a^2}}. \tag{26}$$

These approximate solutions are accurate to within ten percent for ξ less than 0.3 nepers/cavity (2.5 dB/cavity).

In Fig. 7 the experimentally measured attenuation for the "centipede" waveguide is shown for the operating pass band. We estimate $D = 0.0407$ from the midband data and (25). Then, from (23) and (24), we can compute the equivalent circuit attenuation which is also shown in Fig. 7. It can be seen that the circuit provides a very good model of the waveguide attenuation. The difference between the computed curve and the measured data is less than ten percent over the operating pass band.

VI. Conclusions

The equivalent circuit discussed here is the result of a search for a simple model which can represent, at least qualitatively, the propagation characteristics of all coupled-cavity waveguides that use resonant coupling elements. It has been shown that the circuit provides a very good model for the "centipede" waveguide. The dispersion of the two most important pass bands of an S-band "centipede" waveguide has been represented to within a few percent. The interaction ratio and waveguide attenuation for the operating pass band have also been accurately modeled, although for the greatest accuracy it was necessary to allow the circuit elements to vary slightly with frequency.

The most important feature of the equivalent circuit is its simplicity. Only five parameters are needed to specify its propagation characteristics, and these parameters give direct insight into the electrical behavior of the circuit and the analogous waveguide. The parameters f_a, f_b, and k determine the circuit dispersion, C_1 determines the interaction ratio, and D specifies the circuit attenuation. Finally, we note that the circuit represents the important electrical resonances of the waveguide, rather than its mechanical details. The circuit can therefore be expected to provide a qualitative model for other cavity chain waveguides which, like the "centipede," use resonant coupling elements.

Acknowledgment

The author is indebted to Profs. M. Chodorow and G. S. Kino of Stanford University, Stanford, Calif., and to J. Froom of Standard Telecommunication Laboratories, Harlow, Essex, England, for helpful discussions during the course of this work. Thanks are also extended to Miss D. Dickey who carried out the computations.

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Corrections

R. E. Collin, author of the paper, "Electromagnetic Potentials and Field Expansions for Plasma Radiation in Waveguides," which appeared on pages 413–420 of the July, 1965, issue, submits the following.

The upper limit of integration in equations (19) and (24) should be

$$r = t - \frac{|z - z'|}{c}$$

and not infinity.

P. J. Meier and S. Arnow, authors of the paper, "Wide-Band Polarizer in Circular Waveguide Loaded with Dielectric Discs," which appeared on pages 763–767 of the November, 1965, issue, wish to note the following.

In formula (14), page 765, the last term in the denominator ± 1 should appear outside the radical.

In Section II, page 763, λ_2, β_3 should read β_2, β_3.